The kappa function

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Abstract: The kappa function is introduced as the function κ satisfying $J(\kappa(\tau)) = \lambda(\tau)$, where J and λ are the elliptic modular functions. A Fourier expansion of κ is studied.

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1 Introduction

Let G and G' be discrete subgroups of the group $PGL_2(\mathbf{C})$ of linear franctional transformations, and $r : G \to G'$ a surjective homomorphism. A holomorphic function $f(z)$ is said to be *covariant* of type (G, r, G') if

$$f\left(\frac{az+b}{cz+d}\right) = \frac{a'f(z)+b'}{c'f(z)+d'}, \quad \text{for} \quad g = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \in G, \quad r(g) = \left[\begin{array}{cc} a' & b' \\ c' & d' \end{array}\right] \in G'.$$

When G' is trivial, a covariant function is a G-automorphic function. We are interested in the following cases:

1. $G = G'$ is a finite group (r is the identity map). See [OY].

2. $G = G'$ is a triangle Fuchsian group (r is the identity map). An example of covariant functions for $G = PSL_2(\mathbf{Z})$ is given in [KK].

3. G and G' are triangle Fuchsian groups, and $\mathrm{Ker}(r)$ and $G/\mathrm{Ker}(r)$ are both infinite groups.

In this paper, as a typical example of the third case, we introduce the kappa function κ defined by $J(\kappa(\tau)) = \lambda(\tau)$, where J and λ are the elliptic modular functions, and study its Fourier expansion at $i\infty$.

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2 The Schwarz map of the hypergeometric equation

We briefly recall in this section a classical theory of Schwarz maps (cf. [Yos]). Let $E(a,b,c)$ be the hypergeometric differential equation

$$x(1-x)u'' + (c-(a+b+1)x)u' - abu = 0,$$

where a,b and c are parameters. Its Schwarz map is defined by

$$s : X = \mathbf{C} - \{0,1\} \ni x \longmapsto z = u_1(x) : u_2(x) \in Z = \mathbf{P}^1 := \mathbf{C} \cup \{\infty\},$$

where u_1 and u_2 are two linearly independent solutions of $E(a,b,c)$. The local exponents of the equation $E(a,b,c)$ at $0, 1$ and ∞ are given as $\{0, 1-c\}, \{0, c-a-b\}$ and $\{a,b\}$, respectively. Denote the differences of the local exponents by

$$\mu_0 = 1-c, \quad \mu_1 = c-a-b, \quad \mu_\infty = a-b,$$

and the monodromy group by $\mathrm{Monod}(\mu_0,\mu_1,\mu_\infty)$. Then the Schwarzian derivative $\{s;x\}$ of s with respect to x is given as

$$\begin{aligned} -4\{s;x\} &= \frac{2s's''' - 3(s'')^2}{(s')^2} \\ &= \frac{1-\mu_0^2}{x^2} + \frac{1-\mu_1^2}{(1-x)^2} + \frac{1+\mu_\infty^2-\mu_0^2-\mu_1^2}{x(1-x)}. \end{aligned}$$

We assume that the parameters a,b and c are rational numbers such that

$$k_0 := \frac{1}{|\mu_0|}, \quad k_1 := \frac{1}{|\mu_1|}, \quad k_\infty := \frac{1}{|\mu_\infty|} \quad \in \ \{2,3,\dots\} \cup \{\infty\},$$

and $1/k_0 + 1/k_1 + 1/k_\infty < 1$. Then the Schwarz map

$$s = s_{(k_0,k_1,k_\infty)} : X \longrightarrow \mathbf{H} = \{z \in \mathbf{C} \mid \Im(z) > 0\}$$

gives the developing map of the *universal branched covering* with ramification indices (k_0,k_1,k_∞); its inverse map

$$s^{-1} : \mathbf{H} - \mathrm{Fix}(\Delta) \longrightarrow X$$

is single-valued, and induces the isomorphism $(\mathbf{H} - \mathrm{Fix}(\Delta))/\Delta \cong X$, where $\Delta = \Delta_{(k_0,k_1,k_\infty)}$ is the monodromy group $\mathrm{Monod}(k_0,k_1,k_\infty)$ regarded as a transformation group (Schwarz's triangle group) of $\mathbf{H}$, and $\mathrm{Fix}(\Delta)$ is the set of fixed points of Δ.

3 Covariant functions of type $(\Gamma(2), r, \Delta)$

In particular, when $(k_0,k_1,k_\infty) = (\infty,\infty,\infty)$, the monodromy group $\Delta_{(\infty,\infty,\infty)}$ is isomorphic to the principal congruence subgroup

$$\Gamma(2) = \{g \in SL_2(\mathbf{Z}) \mid g \equiv \mathrm{id} \bmod 2\}/\{\pm 1\},$$

which has no fixed points, and the inverse of the Schwarz map is known by the name of the lambda function $\lambda(z)$ defined on $\mathbf{H}$. Since $\lambda : \mathbf{H} \to X$ is the universal covering of X, for any (k_0, k_1, k_∞) satisfying $1/k_0 + 1/k_1 + 1/k_\infty < 1$, the branched covering

$$s^{-1} = s^{-1}_{(k_0,k_1,k_\infty)} : \mathbf{H} - \mathrm{Fix}(\Delta) \longrightarrow X$$

factors λ, that is, there is a unique map $f = f_{(k_0,k_1,k_\infty)} : \mathbf{H} \to \mathbf{H} - \mathrm{Fix}(\Delta)$ such that

$$s^{-1}(f(z)) = \lambda(z), \quad z \in \mathbf{H},$$

where $\Delta = \Delta_{(k_0,k_1,k_\infty)}$. The Galois correspondence can be illustrated as

$$\begin{array}{ccl}
\{1\} & \mathbf{H} & \text{The universal covering of } X \\
| & \downarrow f & \\
N & \mathbf{H} - \mathrm{Fix}(\Delta) & \text{The universal branched covering of } X \\
| & \downarrow s^{-1} & \\
\pi_1(X) & X &
\end{array}$$

Here N is a normal subgroup of the fundamental group $\pi_1(X)$ of X corresponding to the middle cover $\mathbf{H} - \mathrm{Fix}(\Delta)$. Actually, N is given as follows: Let γ_0 (resp. γ_1 and γ_∞) be a simple loop around $x = 0$ (resp. 1 and ∞), and regard these loops as elements of $\pi_1(X)$. Then N is the smallest subgroup of $\pi_1(X)$ containing

$$\gamma_0^{k_0}, \quad \gamma_1^{k_1} \quad \text{and} \quad \gamma_\infty^{k_\infty}.$$

At any rate, we have

$$\pi_1(X)/N \cong \Delta;$$

let $r : \pi_1(X) \to \Delta$ denote the projection.

For a point $z \in \mathbf{H}$, put $w = f(z)$ and $x = \lambda(z)$. Let γ be a loop in X with base x. The lift of γ under λ is a path in $\mathbf{H}$ connecting z and $g(z)$ for some $g \in \Gamma(2)$; this gives the isomorphism

$$\pi_1(X, x) \cong \Gamma(2).$$

The lift of γ under the Schwarz map s is a path in $\mathbf{H} - \mathrm{Fix}(\Delta)$ connecting w and $g'(w)$ for some $g' \in \Delta$; the correspondence

$$\Gamma(2) \ni g \longmapsto g' = r(g) \in \Delta$$

is the homomorphism r via the identification $\pi_1(X, x) \cong \Gamma(2)$.

Proposition 1 *Our function f is covariant of type* $(\Gamma(2), r, \Delta)$.

Proof. We have

$$f(g(z)) = g'(w) = g'(f(z)), \quad g \in \Gamma(2). \qquad \blacksquare$$

The following illustration may help the reader.

$$\begin{array}{ccccc}
\mathbf{H} & \ni & z & \xrightarrow{\lambda^*(\gamma)} & g(z) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{H} - \mathrm{Fix}(\Delta) & \ni & w & \xrightarrow{s_*(\gamma)} & r(g)(w) \\
\downarrow & & \downarrow & & \\
X & \ni & x & \xrightarrow{\gamma} & x
\end{array}$$

3.1 The kappa function

We are especially interested in the case

$$(k_0, k_1, k_\infty) = (3, 2, \infty).$$

The monodromy group $\Delta_{(3,2,\infty)}$ is isomorphic to $\Gamma(1) = PSL_2(\mathbf{Z})$, and the map s^{-1} is usually denoted by J. We name the function f as the *kappa function* κ; this is because the letter k is situated between j and l in the alphabetic sequence. So we have

$$J(\kappa(z)) = \lambda(z).$$

We normalize the maps in question as

$$\begin{array}{ccccccc}
\{1\} & \mathbf{H} & z = & 0 & \infty & 1 \\
| & \downarrow\ \kappa & & \downarrow & \downarrow & \downarrow \\
N & \mathbf{H} - \Gamma(1)\{i, \rho\} & w = & \rho & i & \infty \\
| & \downarrow\ J & & \downarrow & \downarrow & \downarrow \\
\Gamma(2) & X & x = & 0 & 1 & \infty
\end{array}$$

where $\rho = \exp(2\pi i/6)$. Let γ_0 and γ_1 be the simple loops (with base point in the lower half x-plane) around 0 and 1 as are shown in Figure 1. According to the normalization above, γ_0 and γ_1, as elements of $\pi_1(X)$, are identified respectively with the two generators

$$g_0 : z \mapsto \frac{z}{-2z+1} \quad \text{and} \quad g_\infty : z \mapsto z + 2$$

of $\Gamma(2)$; they fix 0 and ∞, respectively. Then the subgroup N is the smallest normal subgroup of $\Gamma(2)$ containing

$$g_0^3 : z \mapsto \frac{z}{-6z+1} \quad \text{and} \quad g_\infty^2 : z \mapsto z + 4,$$

and the isomorphism $\Gamma(2)/N \cong \Gamma(1)$ is given by the surjective homomorphism $r : \Gamma(2) \to \Gamma(1)$ defined by

$$g_0 \longmapsto \left(w \mapsto \frac{1}{1-w}\right) \quad \text{and} \quad g_\infty \longmapsto \left(w \mapsto \frac{-1}{w}\right).$$

Thus our function κ satisfies

$$\kappa\left(\frac{z}{-2z+1}\right) = \frac{1}{1-\kappa(z)} \quad \text{and} \quad \kappa(z+2) = \frac{-1}{\kappa(z)}.$$

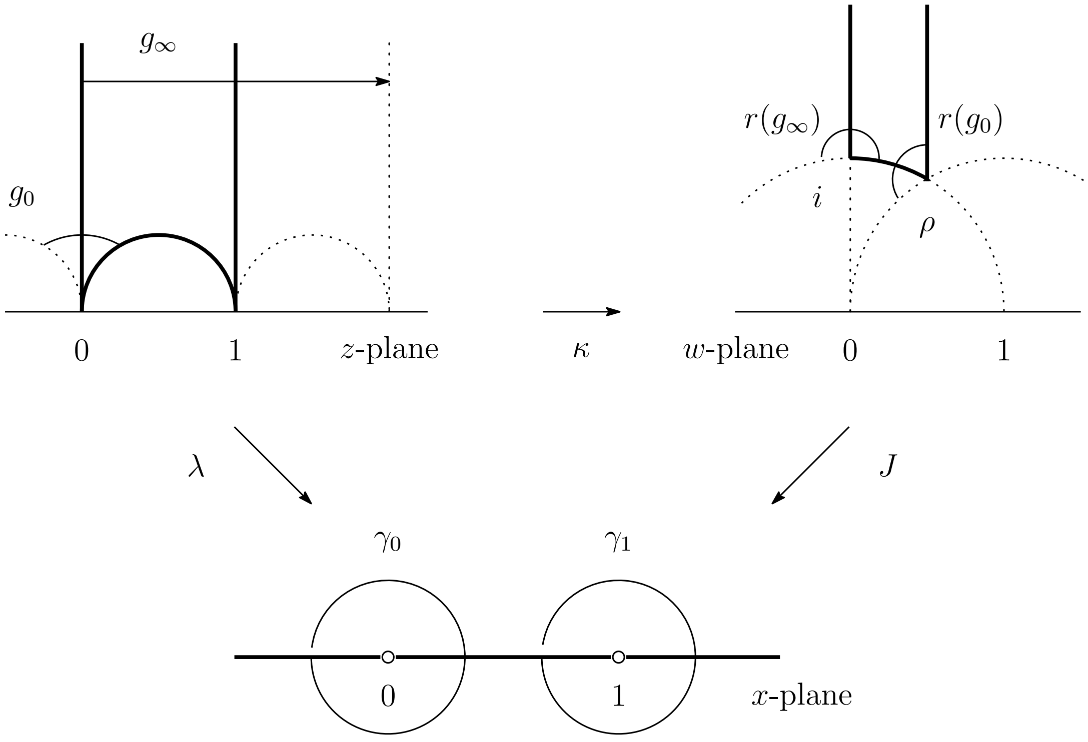

Figure 1: A geometric explanation of the correspondence: $\gamma_0 \leftrightarrow g_0, \gamma_1 \leftrightarrow g_\infty$

3.2 A fundamental domain for N

Recall that the map $\kappa : \mathbf{H} \to \mathbf{H} - \Gamma(1)\{i,\rho\}$ is the universal cover (of the infinitely punctured upper half w-plane $\mathbf{H} - \Gamma(1)\{i,\rho\}$) with the transformation group $N \subset \Gamma(2)$. To obtain a fundamental domain of N in the upper half z-plane, we cut the punctured upper half w-plane so that it becomes simply connected.

Our cut shown in Figure 2 is invariant under the action of $\Gamma(2)$, where $\Gamma(2)$ is here regarded as the *subgroup* of $\Gamma(1)$ acting on the w-space. In the figure, a fundamental domain of $\Gamma(2)$ is shown as the union of twelve triangles $1,\dots,6,\ 1',\dots,6'$, each of which is a fundamental domain of the extended triangle group of $\Gamma(1)$. Our cuts are now given by

$$1\cap 6, \quad 1'\cap 2', \quad 3'\cap 4', \quad 5'\cap 6', \quad 6'\cap 1'.$$

It is easy to check that the complement of the $\Gamma(2)$-orbits of these cuts is connected and simply connected. If we draw this connected net of triangles on the z-plane

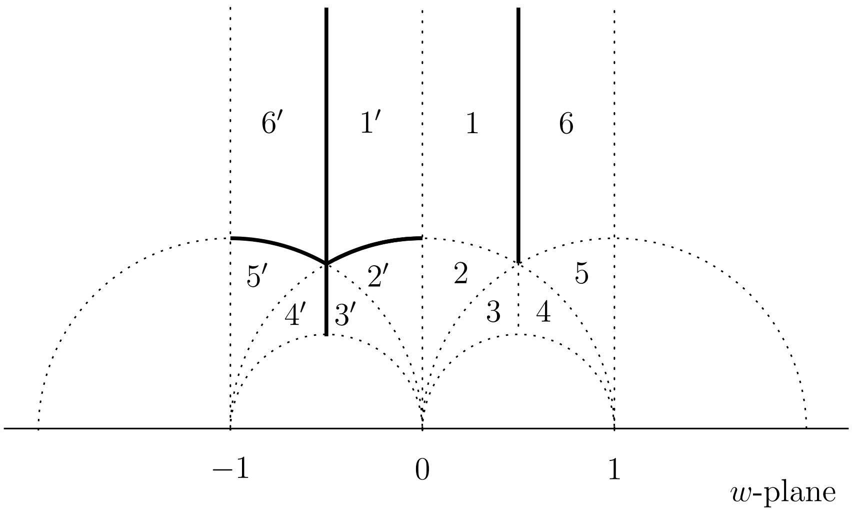

Figure 2: $\Gamma(2)$-invariant cuts of the w-plane $\mathbf{H} - \Gamma(1)\{i, \rho\}$

through κ, shown in Figure 1, making use of the Schwarz reflection principle, we eventually obtain a fundamental domain of N bounded by infinitely many arcs as is shown in Figure 3.

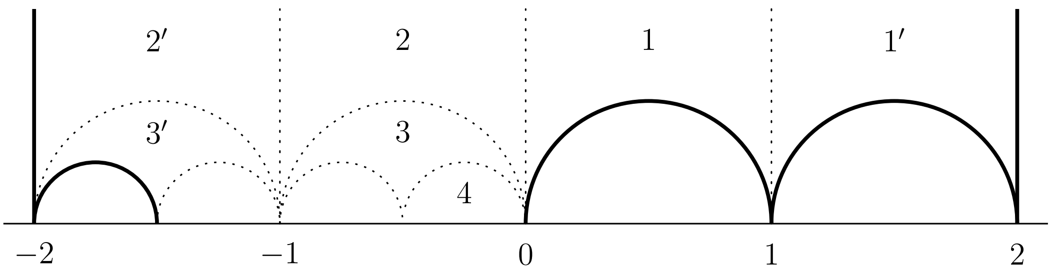

Figure 3: A fundamental domain of N in the z-plane $\mathbf{H}$

3.3 A Fourier expansion of the kappa function

In this section we compute the Fourier development of $\kappa(z)$ at $z = i\infty$. Since $\kappa(z+4) = \kappa(z)$ and $\kappa(\infty) = i$ by definition, the Fourier series of $\kappa(z)$ has the form

$$\kappa(z) = i(1 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots),$$

where

$$q := \exp \frac{\pi i z}{2}.$$

Proposition 2 *1) The nth Fourier coefficient a_n of $\kappa(z)$ can be expressed as a polynomial of degree n in $a := a_1$ with rational coefficients, starting with $a^n/2^{n-1}+\cdots$ and having no constant term. The polynomial is even or odd according as n is even or odd.*

2) The value of a is explicitly given by

$$a = -i\frac{32}{\sqrt{3}}\,\frac{\pi^2}{\Gamma(1/4)^4} = -1.0552729262852\cdots \times i.$$

Example 1

$$\begin{aligned}
a_1 &= a,\\
a_2 &= \frac{1}{2}a^2,\\
a_3 &= \frac{1}{4}a^3 - \frac{16}{27}a,\\
a_4 &= \frac{1}{8}a^4 - \frac{16}{27}a^2,\\
a_5 &= \frac{1}{16}a^5 - \frac{4}{9}a^3 + \frac{98}{1215}a,\\
a_6 &= \frac{1}{32}a^6 - \frac{8}{27}a^4 + \frac{934}{3645}a^2,\\
a_7 &= \frac{1}{64}a^7 - \frac{5}{27}a^5 + \frac{787}{2430}a^3 - \frac{1504}{6561}a,\\
a_8 &= \frac{1}{128}a^8 - \frac{1}{9}a^6 + \frac{41}{135}a^4 - \frac{9088}{32805}a^2.
\end{aligned}$$

Proof. For 1), we shall establish recursion relations among a_n's. First, by the identity

$$\kappa(z+2) = -\frac{1}{\kappa(z)},$$

we immediately obtain the recursion with which the even index coefficients are determined by the previous ones.

Lemma 1 *For each even integer $n \geq 2$, we have*

$$a_n = \sum_{i=1}^{n/2-1}(-1)^{i-1}a_i a_{n-i} + (-1)^{n/2-1}\frac{a_{n/2}^2}{2}. \qquad (1)$$

In particular, $a_2 = a_1^2/2, a_4 = a_1a_3 - a_2^2/2, a_6 = a_1a_5 - a_2a_4 + a_3^2/2, \ldots$.

Proof. Since we have $\kappa(z+2) = i(1 - a_1q + a_2q^2 - a_3q^3 + \cdots)$, we get the recursion by expanding $\kappa(z+2)\kappa(z)$ and equating the coefficient of q^n with 0. ∎

Note $\kappa(z+2)\kappa(z)$ is the even function of q and so for odd n the coefficient is automatically 0. To determine a_n for odd n, we make use of the explicit formula for the Schwarzian derivative $\{\kappa; z\}$. To describe this, we introduce Jacobi's theta constants;

$$\theta_0(z) = \sum_{n\in\mathbf{Z}} (-1)^n q^{2n^2}, \quad \theta_2(z) = \sum_{n\in\mathbf{Z}} q^{\frac{(2n+1)^2}{2}}, \quad \theta_3(z) = \sum_{n\in\mathbf{Z}} q^{2n^2}.$$

They satisfy the famous identity $\theta_0(z)^4 + \theta_2(z)^4 = \theta_3(z)^4$, which will be used later. By these theta's, our λ function can be expressed as

$$\lambda(z) = \frac{\theta_0(z)^4}{\theta_3(z)^4} = 1 - 16q^2 + 128q^4 - 704q^6 + \cdots.$$

In fact, the $\Gamma(2)$-invariance is classical and the only thing we have to check is the values $\lambda(\infty) = 1$, $\lambda(0) = 0$ and $\lambda(1) = \infty$ that we have chosen to normalize λ. But this is readily seen by the above and the following expansions

$$\begin{aligned}
\lambda\left(-\frac{1}{z}\right) &= \frac{\theta_2(z)^4}{\theta_3(z)^4} = 16q^2 - 128q^4 + 704q^6 + \cdots, \\
\lambda\left(1 - \frac{1}{z+1}\right) &= \frac{\theta_3\left(-1/(z+1)\right)^4}{\theta_0\left(-1/(z+1)\right)^4} = \frac{\theta_3(z)^4}{\theta_2(z)^4} = \frac{1}{16q^2} + \frac{1}{2} + \frac{5}{4}q^2 + \cdots,
\end{aligned}$$

which can be derived from the well-known transformation formulae (cf. [Mum])

$$\begin{aligned}
\theta_0(z+1) &= \theta_3(z), & \theta_0(-1/z) &= \sqrt{z/i}\ \theta_2(z), \\
\theta_2(z+1) &= e^{\pi i/4}\theta_2(z), & \theta_2(-1/z) &= \sqrt{z/i}\ \theta_0(z), \\
\theta_3(z+1) &= \theta_0(z), & \theta_3(-1/z) &= \sqrt{z/i}\ \theta_3(z).
\end{aligned}$$

Lemma 2 *We have*

$$\frac{2\kappa'\kappa''' - 3\kappa''^2}{\kappa'^2} = -\frac{1}{9}\left(5\theta_0(z)^4\ \theta_3(z)^4 + 4\theta_3(z)^8\right), \tag{2}$$

where $' = q\dfrac{d}{dq} = \dfrac{2}{\pi i}\dfrac{d}{dz}$.

Proof. Since we have

$$\begin{aligned}
-4\{\lambda^{-1}; x\} &= \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{x(1-x)}, \\
-4\{J^{-1}; x\} &= \frac{1-(1/3)^2}{x^2} + \frac{1-(1/2)^2}{(1-x)^2} + \frac{1-(1/3)^2-(1/2)^2}{x(1-x)},
\end{aligned}$$

the connection formula of the Schwarzian derivative

$$\begin{aligned}
\{\kappa; z\} = \{J^{-1}\circ\lambda; z\} &= \{\lambda; z\} + \{J^{-1}; x\}\left(\frac{dx}{dz}\right)^2 \\
&= \left(-\{\lambda^{-1}; x\} + \{J^{-1}; x\}\right)\left(\frac{dx}{dz}\right)^2
\end{aligned}$$

allows us to express the Schwarzian $\{\kappa; z\}$ as a rational function of $x = \lambda(z)$ and its derivative (we multiply $(2/\pi i)^2$ on both sides to have a formula with $' = q\,d/dq$):

$$\frac{2\kappa'\kappa''' - 3\kappa''^2}{\kappa'^2} = -\frac{\lambda'(z)^2}{36}\left(\frac{5\lambda(z)+4}{\lambda(z)^2(1-\lambda(z))^2}\right).$$

The lemma then follows from the identities

$$\lambda'(z) = -2\theta_2(z)^4\ \lambda(z) \quad \text{and} \quad 1-\lambda(z) = \frac{\theta_2(z)^4}{\theta_3(z)^4}. \qquad \blacksquare$$

Now we use (2) to obtain another recursion for a_n. Put

$$\frac{1}{9}\left(5\theta_0(z)^4\ \theta_3(z)^4 + 4\theta_3(z)^8\right) = \sum_{n=0}^{\infty} b_n q^n.$$

By the formulas

$$\begin{aligned}\theta_0(z)^4\theta_3(z)^4 &= 1 + 16\sum_{n=1}^{\infty}\left(\sum_{d|n}(-1)^d d^3\right) q^{4n},\\ \theta_3(z)^8 &= 1 + 16\sum_{n=1}^{\infty}(-1)^n\left(\sum_{d|n}(-1)^d d^3\right) q^{2n},\end{aligned}$$

the b_n is explicitly given by $b_0 = 1$ and

$$b_n = \begin{cases} 0, & \text{for } n: \text{odd},\\ (-1)^{n/2}\dfrac{64}{9}\displaystyle\sum_{d|n/2}(-1)^d d^3, & \text{for } n \equiv 2 \bmod 4,\\ (-1)^{n/2}\dfrac{64}{9}\displaystyle\sum_{d|n/2}(-1)^d d^3 + \dfrac{80}{9}\displaystyle\sum_{d|n/4}(-1)^d d^3, & \text{for } n \equiv 0 \bmod 4.\end{cases}$$

Equating the coefficients of q^{n+1} on both sides of

$$2\kappa'\kappa''' - 3\kappa''^2 = -\kappa'^2\sum_{n=0}^{\infty} b_n q^n,$$

we obtain, after some manipulation, the recursive relation

$$\begin{aligned}2n(n-1)(n-2)a\cdot a_n = &- \sum_{i=2}^{n-1} i(n+1-i)\left(2(n+1)^2 - 7i(n+1) + 5i^2 + 1\right) a_i a_{n+1-i}\\ &- \sum_{j=1}^{n-1} b_j \sum_{i=1}^{n-j} i(n+1-j-i) a_i a_{n+1-j-i}.\end{aligned}$$

With this recursion and $a_1 = a, a_2 = a^2/2$, we can deduce all the assertions in 1) of Proposition 2 by induction. For parity result we should note that $b_j = 0$ for j odd, and for the top term we use the identity

$$\sum_{i=2}^{n-1} i(n+1-i)\left(2(n+1)^2 - 7i(n+1) + 5i^2 + 1\right) = -2n(n-1)(n-2)$$

and note the second sum on the right has lower degree.

Next we evaluate a. Differentiating the identity $J(\kappa(z)) = \lambda(z)$ twice and multiplying both sides by $\left(\frac{2}{\pi i}\right)^2$, we have

$$\frac{d^2J}{dw^2}(\kappa(z))\left(q\frac{d\kappa}{dq}(z)\right)^2 + \frac{dJ}{dw}(\kappa(z))\left(q\frac{d}{dq}\right)^2\kappa(z) = \left(q\frac{d}{dq}\right)^2\lambda(z) = -64q^2 + \cdots \quad (3)$$

After dividing this by q^2, we look at the limit when $z \to i\infty$ (so $w \to i$ and $q \to 0$). Since

$$\left(q\frac{d\kappa}{dq}(z)\right)^2 = -a^2q^2 + \cdots, \qquad \left(q\frac{d}{dq}\right)^2\kappa(z) = iaq + \cdots,$$

we need the limiting values of $d^2J(w)/dw^2$ and $(dJ(w)/dw)/q$ as $w \to i$ ($w = \kappa(z)$). To compute these, we use the classical Eisenstein series

$$\begin{aligned} E_2(w) &= 1 - 24\sum_{n=1}^{\infty}(\sum_{d|n} d)\, e^{2\pi i n w}, \\ E_4(w) &= 1 + 240\sum_{n=1}^{\infty}(\sum_{d|n} d^3)\, e^{2\pi i n w}, \\ E_6(w) &= 1 - 504\sum_{n=1}^{\infty}(\sum_{d|n} d^5)\, e^{2\pi i n w}, \end{aligned}$$

and the cusp form

$$\Delta(w) = e^{2\pi i w}\prod_{n=1}^{\infty}(1 - e^{2\pi i n w})^{24}.$$

Lemma 3 *We have*

$$\frac{\frac{dJ}{dw}(w)}{q} \longrightarrow -2\pi^2 iaE_4(i) \quad (\textit{as } w \to i)$$

and

$$\frac{d^2J}{dw^2}(i) = -2\pi^2E_4(i).$$

Proof. We use the formula

$$\frac{dE_6}{dw}(w) = \pi i(E_2(w)E_6(w) - E_4(w)^2)$$

as well as the value $E_6(i) = 0$ and $dw/dz = d\kappa(z)/dz = -\pi aq/2 + \cdots$ to obtain (use de L'Hôpital's rule)

$$\lim_{z\to i\infty}\frac{E_6(w)}{q} = \lim_{z\to i\infty}\frac{\pi i\left(E_2(w)E_6(w) - E_4(w)^2\right)\frac{dw}{dz}}{\frac{\pi i}{2}q} = \pi aE_4(i)^2.$$

Hence by

$$\frac{dJ}{dw}(w) = -2\pi i \frac{E_6(w)}{E_4(w)} J(w) \quad \text{and} \quad J(i) = 1,$$

we obtain

$$\lim_{z \to i\infty} \frac{\frac{dJ}{dw}(w)}{q} \longrightarrow -2\pi^2 i a E_4(i).$$

For the second value, we compute

$$\frac{d^2 J}{dw^2}(w) = -2\pi i \left(\frac{d}{dw} \left(\frac{J(w)}{E_4(w)} \right) E_6(w) + \frac{J(w)}{E_4(w)} \cdot \pi i (E_2(w) E_6(w) - E_4(w)^2) \right)$$

and use $E_6(i) = 0, J(i) = 1$. ∎

Applying this lemma to the identity (3) together with the evaluation

$$E_4(i) = \frac{3}{64} \frac{\Gamma(1/4)^8}{\pi^6},$$

we obtain

$$a^2 = -\frac{1024}{3} \frac{\pi^4}{\Gamma(1/4)^8}.$$

Since $\kappa(z)$ tends to i from the right on the unit circle as z goes up to infinity along the pure-imaginary axis, ia must be positive. This proves 2) of Proposion 2. ∎

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